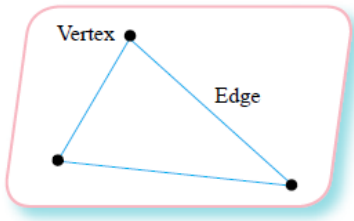
Chapter 5 : Network In Graph Theory

rraezzz

5.1 Network



- Each dot is known as a vertex
- The line joining two vertices is known as an edge.
- A graph is usually used to represent a certain network.
- A network is part of a graph with the vertices and edges having their own characteristics.
- The structure of network data has many-to-many relation.

A graph is denoted by a set of ordered pairs $G = (V, E)$, where

- V is the set of dots or vertices.
  $V = \{v_1, v_2, v_3, \ldots v_n\}$
- E is the set of edges or lines linking each pair of vertices.
  $E = \{e_1, e_2, e_3, \ldots e_n\}$
  $E = \{(a_1, b_1), (a_2, b_2), \ldots (a_n, b_n)\}$; a and b are pairs of vertices.

The **degree**, d, is the number of edges that connect two vertices. The sum of degrees of a graph is twice the number of edges, that is

$$\Sigma d(v) = 2E\,;\, v \in V$$

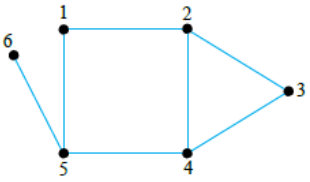
Simple Graph

A simple graph has no loops and no multiple edges. The sum of degrees of the graph is twice the number of edges.

Example 1

Based on the simple graph given, determine

(a) V and $n(V)$
(b) E and $n(E)$
(c) sum of degrees.



Solution:

(a) $V = \{1, 2, 3, 4, 5, 6\}$ ← Set of vertices
$n(V) = 6$ ← Number of vertices

(b) $E = \{(1, 2), (1, 5), (2, 3), (2, 4), (3, 4), (4, 5), (5, 6)\}$ ← Set of vertex pairs
$n(E) = 7$ ← Number of edges

(c) Sum of degrees
$\Sigma d(v) = 2(E)$
$= 2(7)$
$= 14$

Degree of vertex: $d(1) = 2$
$d(2) = 3$
$d(3) = 2$
$d(4) = 3$
$d(5) = 3$
$d(6) = 1$
Sum = 14

The degree of vertex 1 is two, that is the edges which connect vertex 1 to vertex 2 and vertex 1 to vertex 5.

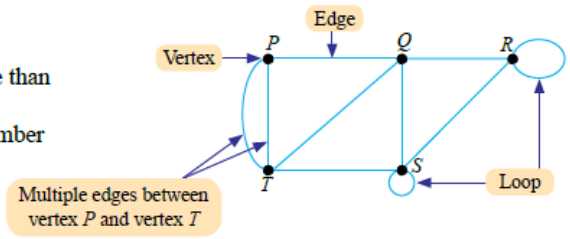
INFO ZONE

The edge for the vertex pair (1, 2) is also the edge for the vertex pair (2, 1).

Multiple edges and loops of a graph

Multiple edges

- Involve two vertices.
- The vertices are connected by more than one edge.
- The sum of degrees is twice the number of edges.



Loops

- Involve one vertex.
- The edge is in the form of an arc that starts and ends at the same vertex.
- Each loop adds 2 to the degree.

Let the graph given be denoted by a set of ordered pairs, $G(V, E)$, then

$V = \{A, B, C, D\}$

Multiple edges

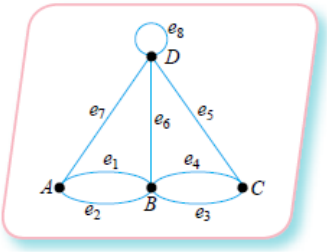
$E = \{\overbrace{(A, B)}, \overbrace{(A, B)}, \overbrace{(B, C)}, \overbrace{(B, C)}, (C, D), (B, D), (A, D), \overbrace{(D, D)}\}$

A loop

$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$

Second edge of AB

First edge of AB



Directed Graph

A **directed graph** is a graph in which a direction is assigned to the edge connecting two vertices. Directed graphs are usually used to represent the flow of a certain process. For example, road maps, airlines networks, electrical circuits, computer networks and organisation charts.

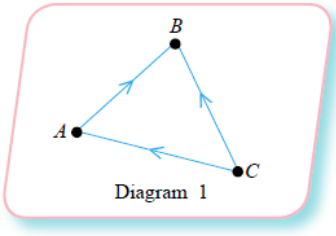


Diagram 1

Diagram 1 shows a simple directed graph. Based on the directions of the arrows, it can be seen that,

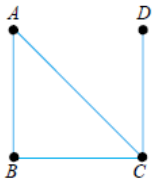
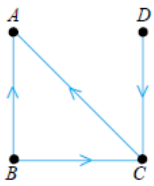
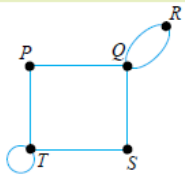
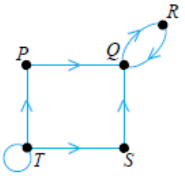
(a) for edge AB, A is the initial vertex and B is the terminal vertex.

(b) for edge CB, C is the initial vertex and B is the terminal vertex.

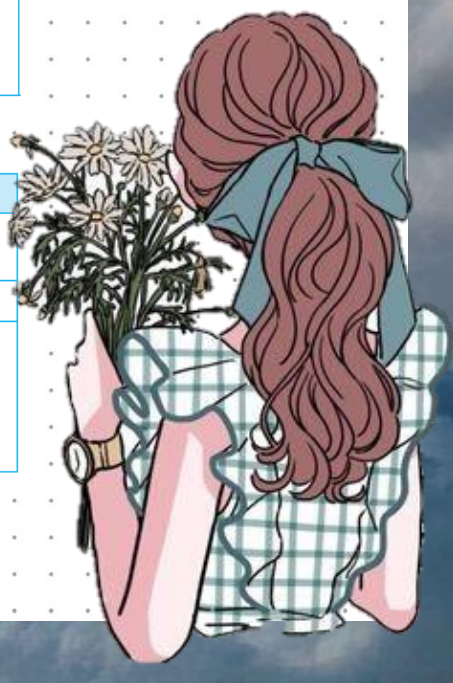
(c) all vertices are connected in one direction only.

rraezzz

Differences between directed graph and undirected graph

Type of graph	Graph	Set V and Set E	Degrees
Simple graph	Undirected graph 	$V = \{A, B, C, D\}$ $E = \{(A, B), (A, C), (B, C), (C, D)\}$ The order of the vertices written is not important. Both pairs of vertices, (A, B) and (B, A) represent the edge AB.	$d(A) = 2$, $d(B) = 2$, $d(C) = 3$, $d(D) = 1$ $\Sigma d(V) = 8$
	Directed graph 	$V = \{A, B, C, D\}$ $E = \{(B, A), (C, A), (B, C), (D, C)\}$ The order of the vertices are written according to the direction of the edge. (B, A) and (A, B) represent different edges.	$d_{in}(A) = 2$ and $d_{out}(A) = 0$ Thus, $d(A) = 2 + 0$ $d(A) = 2$ $d_{in}(A)$ means the number of edges 'going into' vertex A. $d_{out}(A)$ means the number of edges 'coming out' from vertex A. $d_{in}(B) = 0$, $d_{out}(B) = 2$ $d_{in}(C) = 2$, $d_{out}(C) = 1$ $d_{in}(D) = 0$, $d_{out}(D) = 1$ $\Sigma d(V) = 8$
Graph with loops and multiple edges	Undirected graph 	$V = \{P, Q, R, S, T\}$ $E = \{(P, T), (P, Q), (Q, R), (Q, R), (Q, S), (S, T), (T, T)\}$	$d(P) = 2$, $d(Q) = 4$ $d(R) = 2$, $d(S) = 2$ $d(T) = 4$ $\Sigma d(V) = 14$
	Directed graph 	$V = \{P, Q, R, S, T\}$ $E = \{(P, Q), (Q, R), (R, Q), (S, Q), (T, S), (T, T), (T, P)\}$	$d_{in}(P) = 1$, $d_{out}(P) = 1$ $d_{in}(Q) = 3$, $d_{out}(Q) = 1$ $d_{in}(R) = 1$, $d_{out}(R) = 1$ $d_{in}(S) = 1$, $d_{out}(S) = 1$ $d_{in}(T) = 1$, $d_{out}(T) = 3$ Loop = 1 Loop = 1 $(T, P) = 1$ $(T, S) = 1$ $\Sigma d(V) = 14$

	Weighted graph	Unweighted graph
Type of graph	Directed graph and undirected graph	Directed graph and undirected graph
Edge	Associated with a value or a weight	Not associated with a value or a weight
Example	The edge represents: • distance between two cities. • travelling time. • the current in an electrical circuit. • cost.	The edge relates information like: • job hierarchy in an organisation chart. • flow map. • tree map. • bubble map.



rraezzz

Subgraph

A **subgraph** is part of a graph or the whole graph redrawn without changing the original positions of the vertices and edges. A graph H is said to be a subgraph of G if,

Identify and draw subgraphs and trees.

(a) the vertex set of graph H is a subset of the vertex set of graph G, that is $V(H) \subset V(G)$.
(b) the edge set of graph H is a subset of the edge set of graph G, that is $E(H) \subset E(G)$.
(c) the vertex pairs of the edges of graph H are the same as the edges of graph G.

In short,

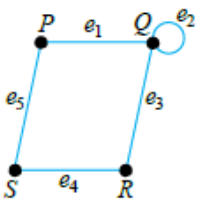
- a vertex in graph G is a subgraph of graph G.
- an edge in graph G along with the vertices it connects is a subgraph of graph G.
- each graph is a subgraph of itself.

INFO ZONE

The symbol $\subseteq$ also stands for subset.

Example 9

Determine whether Diagram 1, Diagram 2, Diagram 3 and Diagram 4 are the subgraphs of graph G.



Graph G

Diagram 1

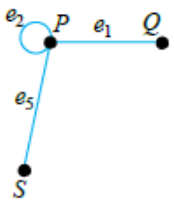


Diagram 2

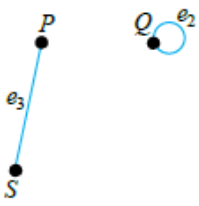


Diagram 3

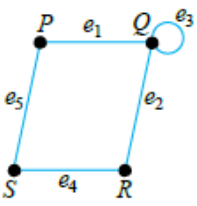


Diagram 4

Solution:

Diagram 1 – Yes, because the vertex pair for edge e_5 is the same.
$\{e_5\} \subset \{e_1, e_2, e_3, e_4, e_5\}$ and $\{P, S\} \subset \{P, Q, R, S\}$

Diagram 2 – No, because the position of loop e_2 is not on vertex Q.

Diagram 3 – No, because the edge connecting vertex P and vertex S is not e_3.

Diagram 4 – No, because the loop and the edge connecting vertex Q and vertex R are wrong.

Tree

A tree of a graph is a subgraph of the graph with the following properties:

(a) A simple graph without loops and multiple edges.

(b) All the vertices are connected and each pair of vertices is connected by only one edge.

(c) Number of edges = Number of vertices − 1
Number of vertices = n
Number of edges = $n - 1$

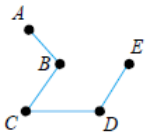


Diagram 1

Diagram 1 is a tree because
- all the vertices are connected.
- every pair of vertices is connected by an edge only.
- there are no loops or multiple edges.
- 5 vertices, 4 edges.

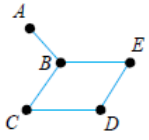
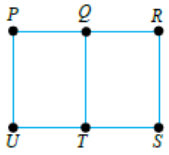


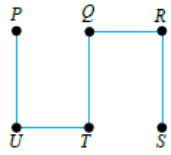
Diagram 2

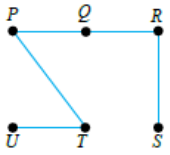
Diagram 2 is not a tree because
- vertex B and vertex E can be connected in two ways.
  (i) $B \to E$
  (ii) $B \to C \to D \to E$
- 5 vertices, 5 edges.

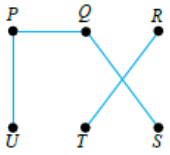
Example 10

Determine whether each of the following graphs is a tree. Justify your answer.

(a) 

(b) 

(c) 

(d) 

Solution:

(a) Not a tree. 6 vertices, 7 edges. Each pair of vertices can be connected in various ways.

(b) A tree. 6 vertices, 5 edges. Each pair of vertices is connected by one edge.

(c) A tree. 6 vertices, 5 edges. Each pair of vertices is connected by one edge.

(d) Not a tree. 6 vertices, 4 edges. Vertex R and vertex T are not connected to the other vertices.

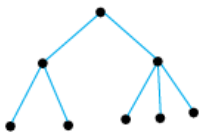
Example 11

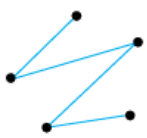
Draw a tree for the following information given.

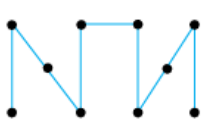
(a) 6 vertices (b) 8 vertices (c) 4 edges (d) 9 edges

Solution:

(a) 6 vertices
5 edges

(b) 8 vertices
7 edges


(c) 4 edges
5 vertices


(d) 9 edges
10 vertices


raezzz